

# A Topological Representation Theorem for Tropical Oriented Matroids: Part II

Silke Horn

TU Darmstadt

shorn@opt.tu-darmstadt.de

December 11, 2012

Tropical oriented matroids were defined by Ardila and Develin in 2007. They are a tropical analogue of classical oriented matroids in the sense that they encode the properties of the types of points in an arrangement of tropical hyperplanes – in much the same way as the covectors of (classical) oriented matroids describe the types in arrangements of linear hyperplanes.

Ardila and Develin proved that tropical oriented matroids can be represented as mixed subdivisions of dilated simplices. In this paper we show that this correspondence is a bijection. Moreover, a tropical analogue for the Topological Representation Theorem for (classical) oriented matroids by Folkman and Lawrence is presented.

## 1 Introduction

Oriented matroids abstract the combinatorial properties of arrangements of real hyperplanes and are ubiquitous in combinatorics. In fact, an arrangement of  $n$  (oriented) real hyperplanes in  $\mathbb{R}^d$  induces a regular cell decomposition of  $\mathbb{R}^d$ . Then the covectors of the associated oriented matroid encode the position of the points of  $\mathbb{R}^d$  (respectively, the cells in the subdivision) relative to the each of the hyperplanes in the arrangement. It turns out though that there are oriented matroids which cannot be realised by any arrangement of hyperplanes. The famous Topological Representation Theorem by Folkman and Lawrence [FL78] (see also [BLS+99]), however, states that every oriented matroid can be realised as an arrangement of PL-*pseudohyperplanes*.

In this paper, we will study *tropical* analogues of oriented matroids.

Tropical geometry is a by now well established subject, see *e.g.* [AB07; AK06; DS04; Mik06]. It is concerned with the algebraic geometry over the tropical semiring  $(\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}, \oplus, \otimes)$ , where  $\oplus : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \rightarrow \bar{\mathbb{R}} : a \oplus b := \min\{a, b\}$  and  $\otimes : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \rightarrow \bar{\mathbb{R}} : a \otimes b := a + b$  are the tropical addition and multiplication. It can be thought of as the image of a field

of formal Puiseux series under the valuation map which takes a power series to its smallest exponent.

From the combinatorial point of view though a tropical hyperplane in  $\mathbb{T}^{d-1}$  is just the (codimension-1-skeleton of the) polar fan of the  $(d-1)$ -dimensional simplex  $\Delta^{d-1}$ . For a  $(d-2)$ -dimensional tropical hyperplane  $H$  the  $d$  connected components of  $\mathbb{T}^{d-1} \setminus H$  are called the *(open) sectors* of  $H$ .

An arrangement of  $n$  tropical hyperplanes in  $\mathbb{T}^{d-1}$  induces a cell decomposition of  $\mathbb{T}^{d-1}$  and each cell can be assigned a *type* that describes its position relative to each of the tropical hyperplanes. To be precise, the point  $p$  is assigned the type  $A = (A_1, \dots, A_n)$  where  $A_i$  denotes the set of closed sectors of the  $i$ -th tropical hyperplane in which  $p$  is contained. See Figure 1(c) for an illustration in dimension 2.

It turns out that tropical curves – and as such in particular arrangements of tropical hyperplanes – have relationships to other interesting objects. Triangulations of products of two simplices are ubiquitous and utile objects in discrete geometry due to their connection with toric Hilbert schemes [San05a] and Schubert calculus [AB07] among others.

By Develin and Sturmfels [DS04] *regular* subdivisions of  $\Delta^{n-1} \times \Delta^{d-1}$  are dual to arrangements of  $n$  tropical hyperplanes in  $\mathbb{T}^{d-1}$ . See Figure 1 for an illustration.

A central concept is that of an  $(n, d)$ -type.

**Definition 1.1.** For  $n, d \geq 1$  an  $(n, d)$ -type is an  $n$ -tuple  $(A_1, \dots, A_n)$  of non-empty subsets of  $[d]$ .

For convenience we will write sets like  $\{1, 2, 3\}$  as 123 throughout this article.

An  $(n, d)$ -type  $A$  can be represented as a subgraph  $K_A$  of the complete bipartite graph  $K_{n,d}$ : Denote the vertices of  $K_{n,d}$  by  $N_1, \dots, N_n, D_1, \dots, D_d$ . Then the edges of  $K_A$  are  $\{\{N_i, D_j\} \mid j \in A_i\}$ .

Besides tropical hyperplane arrangements there are other objects that share the notion of an  $(n, d)$ -type:

- If we label the vertices of  $\Delta^{n-1}$  by  $1, \dots, n$ , the vertices of the polytope  $\Delta^{n-1} \times \Delta^{d-1}$  are in canonical bijection with the edges of the complete bipartite graph  $K_{n,d}$ . Then a cell  $C$  in a subdivision of  $\Delta^{n-1} \times \Delta^{d-1}$  is assigned the type corresponding to the subgraph of  $K_{n,d}$  containing all edges that mark vertices of  $C$ . See *e.g.* De Loera, Rambau and Santos [DRS10] for a thorough treatment of this matter.
- Given a mixed subdivision of  $n\Delta^{d-1}$ , every cell is a Minkowski sum of  $n$  faces of  $\Delta^{d-1}$ . By identifying the faces of  $\Delta^{d-1}$  with the subsets of  $[d]$ , this again yields an  $(n, d)$ -type. See Figure 1(a) for an example. We introduce mixed subdivisions in Section 3.
- Tropical oriented matroids as defined by Ardila and Develin [AD09] via a set of covector axioms generalise tropical hyperplane arrangements. We define them in Section 2.

We continue by briefly pointing out what is known about the relations between the above objects. By the Cayley Trick (*cf.* Huber, Rambau and Santos [HRS00]) subdivisions of  $\triangle^{n-1} \times \triangle^{d-1}$  are in bijection with mixed subdivisions of  $n\triangle^{d-1}$ .

By [AD09, Theorem 6.3], the types of a tropical oriented matroid with parameters  $(n, d)$  yield a subdivision of  $\triangle^{n-1} \times \triangle^{d-1}$ . They also conjecture this to be a bijection. By [AD09, Proposition 6.4], these types satisfy all but one of the tropical oriented matroid axioms.

In Oh and Yoo [OY11] it is proven that *fine* mixed subdivisions satisfy the elimination axiom.

Moreover, [H12b] provides further evidence for the close relationship between mixed subdivisions of  $n\triangle^{d-1}$  and tropical oriented matroids, respectively arrangements of tropical hyperplanes. *E.g.* [H12b, Theorem 4.2] shows that the Poincaré dual of a mixed subdivision of  $n\triangle^{d-1}$  is a family of tropical pseudohyperplanes.

In this paper we introduce arrangements of tropical pseudohyperplanes and prove a tropical analogue to the Topological Representation Theorem for (classical) oriented matroids by Folkman and Lawrence [FL78]. Another variant of the Topological Representation Theorem for a different definition of tropical pseudohyperplane arrangements is contained in [H12b].

A *tropical pseudohyperplane* is basically a set which is PL-homeomorphic to a tropical hyperplane (see also Definition 5.1). The challenging part is the definition of arrangements of these: We have to impose restrictions on the intersections of the pseudohyperplanes in the arrangement. In the classical framework, the intersections of the hyperplanes in the arrangement have to be homeomorphic to linear hyperplanes (of smaller dimension). In the tropical world, however, this approach is not feasible, since intersections of tropical hyperplanes are no longer homeomorphic to tropical hyperplanes (but have a very complicated geometry). In [H12b], we instead imposed restrictions on the cell decomposition induced by the tropical pseudohyperplanes in the arrangement. Here we choose yet another approach that is conceptually closer to the classical case.

An family of tropical pseudohyperplanes is an arrangement if any set of tropical half-

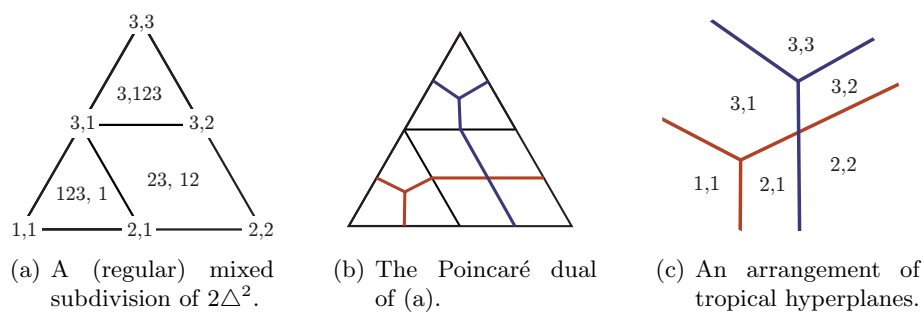


Figure 1: The correspondence between mixed subdivisions and tropical pseudohyperplane arrangements.

space boundaries forms an arrangement of affine pseudohyperplanes.

With this definition we prove the Topological Representation Theorem:

**Theorem 1.2** (Topological Representation Theorem). *Every tropical oriented matroid (in general position) can be realised by an arrangement of tropical pseudohyperplanes.*

We also introduce a theory of combinatorial tropical convexity that is closely related to the elimination property of tropical oriented matroids. In fact, it turns out that a mixed subdivision of  $n\Delta^{d-1}$  satisfies the elimination property if and only if the combinatorial convex hull of any two cells is path-connected. Since any intersection of affine halfspaces is path-connected, we obtain the following application of Theorem 1.2:

We show that *all* mixed subdivisions of  $n\Delta^{d-1}$  satisfy the elimination property and hence prove the conjecture by Ardila and Develin:

**Theorem 1.3** (Cf. [AD09, Conjecture 5.1]). *Tropical oriented matroids with parameters  $(n, d)$  are in bijection with subdivisions of  $\Delta^{n-1} \times \Delta^{d-1}$  and mixed subdivisions of  $n\Delta^{d-1}$ .*

For quick reference, the general picture is depicted in Figure 2.

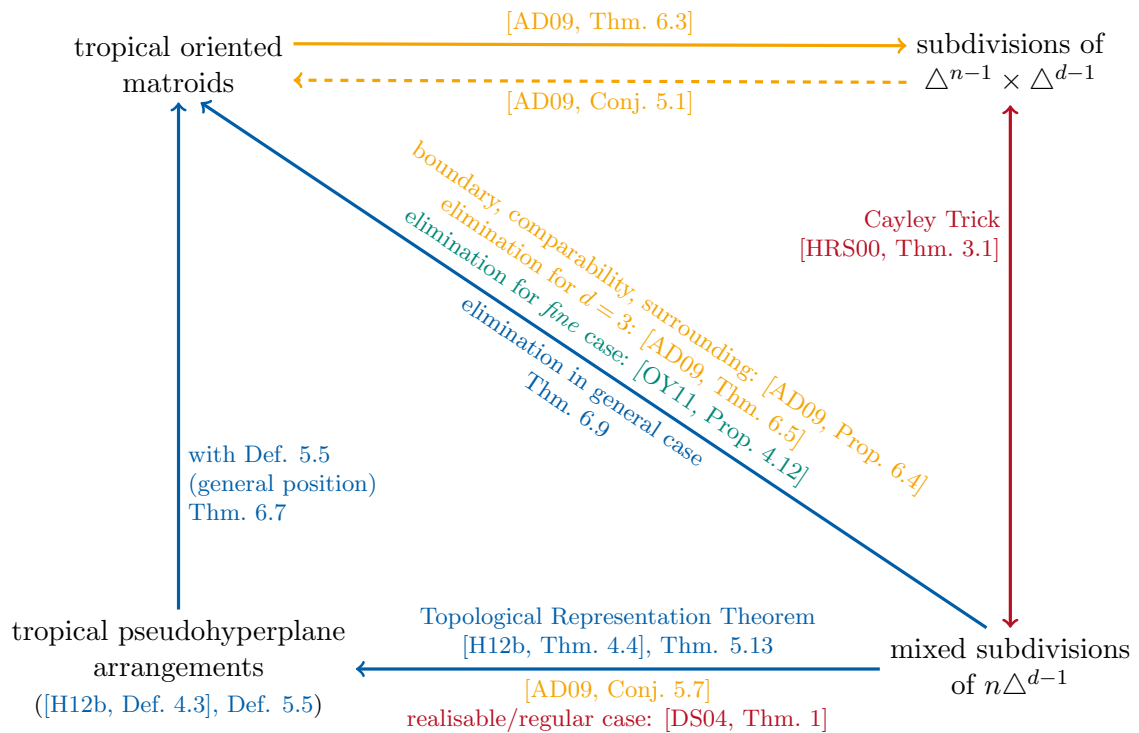


Figure 2: The correspondences between the four concepts of tropical oriented matroids, mixed subdivisions of  $n\Delta^{d-1}$ , subdivisions of a product of two simplices and tropical pseudohyperplane arrangements.

The paper is organised as follows: In Section 2 we briefly review the definition of tropical oriented matroids. In Section 3 we discuss mixed subdivisions of dilated simplices. In Section 4 we have a closer look at the elimination property and define a notion of convexity in tropical oriented matroids. In Section 5 we introduce arrangements of tropical pseudohyperplanes in analogy to (classical) pseudohyperplane arrangements (see Definition 5.5) and prove a Topological Representation Theorem (Theorem 5.13). Finally, in Section 6 we apply our results to prove Theorem 1.3.

This is the follow-up paper of [H12b]. A joint extended abstract [H12a] of this and [H12b] has been presented at FPSAC 2012. Moreover, the results are also contained in [H12d].

## 2 Tropical Oriented Matroids

The following definitions are analogous to those in [AD09], respectively [H12b].

A *refinement* of an  $(n, d)$ -type  $A$  with respect to an ordered partition  $P = (P_1, \dots, P_k)$  of  $[d]$  is the  $(n, d)$ -type  $B = A|_P$  where  $B_i = A_i \cap P_{m(i)}$  and  $m(i)$  is the smallest index where  $A_i \cap P_{m(i)}$  is non-empty for each  $i \in [n]$ . A refinement is *total* if all  $B_i$  are singletons.

Given  $(n, d)$ -types  $A$  and  $B$ , the *comparability graph*  $\mathbb{G}_{A,B}$  is a multigraph with node set  $[d]$ . For  $1 \leq i \leq n$  there is an edge for every  $j \in A_i, k \in B_i$ . This edge is undirected if  $j, k \in A_i \cap B_i$  and directed  $j \rightarrow k$  otherwise. (We consider the comparability graph as a graph without loops.) Note that there may be several edges (with different directions) between two nodes.

A *directed path* in the comparability graph is a sequence  $e_1, e_2, \dots, e_k$  of incident edges at least one of which is directed and all directed edges of which are directed in the “right” direction. A *directed cycle* is a directed path whose starting and ending point agree. The graph is *acyclic* if it contains no directed cycle.

**Definition 2.1** (Cf. [AD09, Definition 3.5]). A *tropical oriented matroid*  $M$  (with parameters  $(n, d)$ ) is a collection of  $(n, d)$ -types which satisfies the following four axioms:

- *Boundary*: For each  $j \in [d]$ , the type  $(j, j, \dots, j)$  is in  $M$ .
- *Comparability*: The comparability graph  $\mathbb{G}_{A,B}$  of any two types  $A, B \in M$  is acyclic.
- *Elimination*: If we fix two types  $A, B \in M$  and a position  $j \in [n]$ , then there exists a type  $C$  in  $M$  with  $C_j = A_j \cup B_j$  and  $C_k \in \{A_k, B_k, A_k \cup B_k\}$  for  $k \in [n]$ .
- *Surrounding*: If  $A$  is a type in  $M$ , then any refinement of  $A$  is also in  $M$ .

We call  $d =: \text{rank } M$  the *rank* and  $n$  the *size* of  $M$ .

**Example 2.2.** By [AD09, Theorem 3.6] the set of types of an arrangement of  $n$  tropical hyperplanes in  $\mathbb{T}^{d-1}$  is a tropical oriented matroid with parameters  $(n, d)$ .

We call tropical oriented matroids coming from an arrangement of tropical hyperplanes *realisable*. Recall that by Develin and Sturmfels [DS04] realisable tropical oriented matroids are in bijection with *regular* mixed subdivisions of  $n\Delta^{d-1}$ .

**Definition 2.3.** The *dimension* of an  $(n, d)$ -type  $A$  is the number of connected components of  $K_A$  minus 1. A *vertex* is a type of dimension 0, an *edge* a type of dimension 1 and a *tope* a type of full dimension  $d - 1$ , *i.e.*, each tope is an  $n$ -tuple of singletons.

A tropical oriented matroid  $M$  is *in general position* if for every type  $A \in M$  the graph  $K_A$  is acyclic.

For two types  $A, B$  we write  $A \supseteq B$  if  $A_i \supseteq B_i$  for each  $i \in [n]$ .

**Definition 2.4** (Cf. [AD09, Propositions 4.7 and 4.8]). Let  $M$  be a tropical oriented matroid with parameters  $(n, d)$ .

1. For  $i \in [n]$  the *deletion*  $M_{\setminus i}$  consisting of all  $(n - 1, d)$ -types which arise from types of  $M$  by deleting coordinate  $i$  is a tropical oriented matroid with parameters  $(n - 1, d)$ .
2. For  $j \in [d]$  the *contraction*  $M_{/j}$  consisting of all types of  $M$  that do not contain  $j$  in any coordinate is a tropical oriented matroid with parameters  $(n, d - 1)$ .

There is also a notion of duality for  $(n, d)$ -types:

**Definition 2.5** (Cf. [AD09, Definitions 5.3 and 5.4]). If  $A$  is a bounded  $(n, d)$ -type then we get a  $(d, n)$ -type  $A^T$ , the *dual type* of  $A$ , by interchanging the roles of  $n$  and  $d$  in the type graph  $K_A$ ; *i.e.*,  $A^T$  is defined by

$$i \in A_j \iff j \in A_i^T.$$

If  $M$  is a tropical oriented matroid with parameters  $(n, d)$  then we define the *dual*  $M^T$  by

$$M^T := \{A^T|_P \mid A \text{ vertex of } M, p \text{ ordered partition of } [n]\}.$$

We will later see in Corollary 6.11 that if  $M$  is a tropical oriented matroid with parameters  $(n, d)$ , then its dual  $M^T$  is a tropical oriented matroid with parameters  $(d, n)$ .

### 3 Mixed Subdivisions of $n\Delta^{d-1}$

Given two sets  $X, Y$  their *Minkowski sum*  $X + Y$  is given by  $X + Y := \{x + y \mid x \in X, y \in Y\}$ .

**Definition 3.1.** Let  $P_1, \dots, P_k \subset \mathbb{R}^n$  be (full-dimensional) convex polytopes. Then a polytopal subdivision  $\{Q_1, \dots, Q_s\}$  of  $P := \sum P_i$  is a *mixed subdivision* if it satisfies the following conditions:

1. Each  $Q_i$  is a Minkowski sum  $Q_i = \sum_{j=1}^k F_{i,j}$ , where  $F_{i,j}$  is a face of  $P_j$ .

2. For  $i, j \in [s]$  we have that  $Q_i \cap Q_j = (F_{i,1} \cap F_{j,1}) + \dots + (F_{i,k} \cap F_{j,k})$ .

A mixed subdivision of  $n\Delta^{d-1}$  is *fine* if there is no other mixed subdivision of  $n\Delta^{d-1}$  refining it.

We are interested in the case of mixed subdivisions where  $P_i = \Delta^{d-1}$  for each  $i$ . Then  $\sum P_i = n\Delta^{d-1}$  is a dilated simplex. By Santos [San05b] a subdivision of  $n\Delta^{d-1}$  is mixed if and only if each cell is a Minkowski sum of  $n$  faces of  $\Delta^{d-1}$ . By Ardila and Develin [AD09, Theorem 6.3] the types of a tropical oriented matroid with parameters  $(n, d)$  yield a mixed subdivision of  $n\Delta^{d-1}$ . A tropical oriented matroid is in general position if and only if its mixed subdivision is fine.

To avoid confusion with the vertices of tropical oriented matroids, we speak of the 0-dimensional cells of a mixed subdivision as *topes*. By [H12b, Proposition 3.1], a mixed subdivision of  $n\Delta^{d-1}$  is uniquely determined by its topes.

### 3.1 Placing in mixed subdivisions

Recall that triangulations of  $\Delta^{n-1} \times \Delta^{d-1}$  are in bijection with the fine mixed subdivisions of  $n\Delta^{d-1}$  via the Cayley Trick. There is a well-known construction that produces a triangulation of  $\Delta^{n'} \times \Delta^{d'}$  (called the *placing triangulations*) from one of  $\Delta^n \times \Delta^d$  for  $n' \geq n, d' \geq d$ . See De Loera, Rambau and Santos [DRS10, Section 4.3.1] for more details.

Since we will need this construction in Section 6, we now examine how placing works in the mixed subdivision point of view:

Suppose we are given a mixed subdivision  $S$  of  $n\Delta^{d-1}$ . Let  $T$  be the corresponding subdivision of  $\Delta^{n-1} \times \Delta^{d-1}$ . There are two possible ways to extend this by placing:

- We can embed  $T$  into  $\Delta^n \times \Delta^{d-1}$ . *I.e.*, we extend  $S$  to a mixed subdivision of  $(n+1)\Delta^{d-1}$ .
- We can embed  $T$  into  $\Delta^{n-1} \times \Delta^d$ . *I.e.*, we extend  $S$  to a mixed subdivision of  $n\Delta^d$ .

We will call the operations *n-placing*, respectively *d-placing*, referring to whether we increase  $n$  or  $d$ . The two operations are of course dual to each other.

**n-Placing** There are  $d$  vertices to be placed, namely the vertices  $(n+1, 1), \dots, (n+1, d)$ . We will denote both the mixed subdivision of  $n\Delta^{d-1}$  and the corresponding subdivision of  $\Delta^{n-1} \times \Delta^{d-1}$  by  $S$ . Moreover, we will apply operations as defined for tropical oriented matroids to the types of both mixed subdivisions and triangulations of products of simplices.

Let  $\sigma$  be some permutation of  $[d]$ . First we place the vertex  $(n+1, \sigma_1)$ . From this vertex every maximal (*i.e.*,  $(n+d-2)$ -dimensional) simplex of  $S$  is visible. Thus, for every maximal simplex  $B$  we add the simplex  $B \cup \{\sigma_1\}$  and all its faces to  $S$  to get  $S_1$ . In the mixed subdivision this corresponds to adding a new entry  $\{\sigma_1\}$  at the end of every type in  $S$ . Thus,  $S_1$  is just a copy of  $S$  in the  $\sigma_1$ -th corner of  $(n+1)\Delta^{d-1}$ .

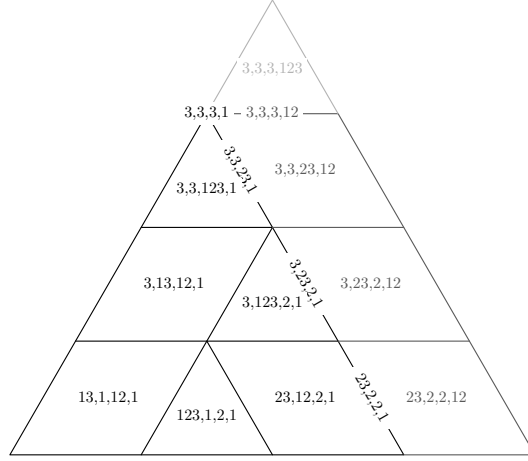


Figure 3: A mixed subdivision  $S$  of  $3\Delta^2$  (black) in its  $n$ -placing extension with respect to the permutation  $(1, 2, 3)$ .

As for placing the vertex  $(n+1, \sigma_2)$ , the only visible simplices are those whose type does not contain  $\sigma_1$  except in the last entry (where we just added it). In the mixed subdivision, placing  $(n+1, \sigma_2)$  corresponds to appending a new entry  $\{\sigma_1, \sigma_2\}$  to the end of every vertex in the contraction  $S_{/\sigma_1}$  and then adding all refinements of those to obtain  $S_2$ .

Placing the remaining vertices works similarly: When placing  $(n+1, \sigma_i)$ , we create the set  $S_i$  containing all vertices in the contraction  $S_{/\{\sigma_1, \dots, \sigma_{i-1}\}}$  with a new entry  $\{\sigma_1, \dots, \sigma_i\}$  appended and all refinements of those.

Figure 3 shows an example of an  $n$ -placing extension.

**$d$ -Placing** There are  $n$  vertices to be placed, namely the vertices  $(1, d+1), \dots, (n, d+1)$ .

Let  $\tau$  be some permutation of  $[n]$ . Recall that for the construction of the  $n$ -placing extension the contractions  $S_{/\sigma_1}, S_{/\{\sigma_1, \sigma_2\}}, \dots, S_{/\{\sigma_1, \sigma_2, \dots, \sigma_d\}}$  for some permutation  $\sigma$  of  $[d]$  played an important role.

In the same way, the deletions  $S_{\setminus \tau_1}, S_{\setminus \{\tau_1, \tau_2\}}, \dots, S_{\setminus \{\tau_1, \tau_2, \dots, \tau_n\}}$  will be important in the construction of the  $d$ -placing extension of  $S$ .

We will only consider the maximal simplices in  $S'$ .

First place the vertex  $(\tau_1, d+1)$ . From this vertex every maximal simplex in  $S$  is visible. Hence for every maximal simplex  $B$  we add the simplex  $B \cup \{(\tau_1, d+1)\}$  to get  $S_1$ . In the mixed subdivision this corresponds to adding  $d+1$  to  $B_{\tau_1}$ .

When we then place  $(\tau_2, d+1)$ , the visible simplices are the simplices in  $S$  with the  $\tau_i$ -th entry replaced by  $\{d+1\}$ .

In general, when placing the  $i$ -th vertex  $(\tau_i, d+1)$ , the visible simplices correspond to the cells in the deletion  $S_{\setminus \{\tau_1, \dots, \tau_{i-1}\}}$  with additional entries  $\{d+1\}$  at the positions  $\tau_1, \dots, \tau_{i-1}$ .

See Figure 4 for an illustration.



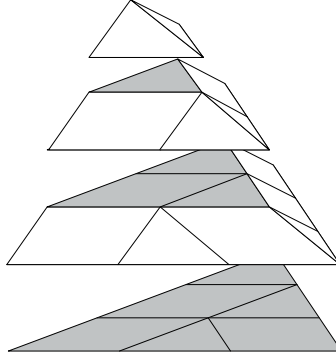


Figure 4: A 3-dimensional  $d$ -placing extension of a mixed subdivision of  $3\Delta^2$ .

## 4 Convexity in tropical oriented matroids and the elimination property

Recall that by Ardila and Develin [AD09, Theorem 6.3] the types of a tropical oriented matroid with parameters  $(n, d)$  yield a subdivision of  $\Delta^{n-1} \times \Delta^{d-1}$ . Since by [AD09, Proposition 6.4] these types satisfy the boundary, comparability and surrounding axioms, the only thing left open is elimination.

By Oh and Yoo [OY11, Proposition 4.6], *fine* mixed subdivisions satisfy the elimination property.

In the realisable case, the *elimination axiom* describes the intersection of a tropical line segment from  $A$  to  $B$  with the  $j$ -th tropical hyperplane. In other words, in the according arrangement of tropical *pseudohyperplanes* (dual to the mixed subdivision) all eliminations of  $A$  and  $B$  (for all  $j$ ) describe the line segment from  $A$  to  $B$ .

One can exploit the elimination property of tropical oriented matroids to obtain topological properties of the according mixed subdivisions.

**Definition 4.1.** Let  $M$  be a tropical oriented matroid and  $A, B \in M$  two types. Then the set

$$M_{AB} := \{C \in M \mid C_i \in \{A_i, B_i, A_i \cup B_i\} \text{ for all } i \in [n]\}$$

is the (*combinatorial*) *convex hull* of  $A$  and  $B$ . Analogously we define the (*combinatorial*) *convex hull*  $S_{AB}$  of two cells in a mixed subdivision  $S$  of  $n\Delta^{d-1}$ .

We say that a subset  $C$  of a tropical oriented matroid  $M$  (or equivalently, a subcomplex of a mixed subdivision of  $n\Delta^{d-1}$ ) is *convex* if for any  $A, B \in C$  we have that  $M_{AB} \subseteq C$ .

Develin and Sturmfels [DS04] defined a notion of convexity in tropical geometry: Given two points  $x, y \in \mathbb{T}^{d-1}$  the *tropical line segment* connecting them is the set

$$[x, y]_{\text{trop}} := \{(\lambda \otimes x) \oplus (\mu \otimes y) \mid \lambda, \mu \in \mathbb{R}\}.$$

The above notion for convexity in tropical oriented matroids generalises this in a natural way: In the realisable case the convex hull  $M_{AB}$  of two types contains all cells

that intersect a tropical line segment between two points in open cells of types  $A$  and  $B$  in *some* realisation of  $M$ . See Figure 5 for an illustration.

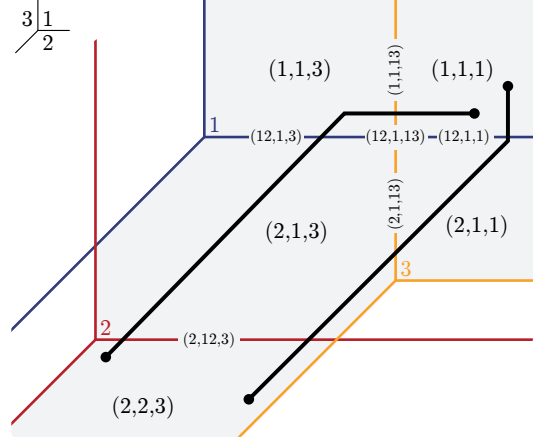


Figure 5: The convex hull of two types  $A = (2, 2, 3)$ ,  $B = (1, 1, 1)$  in a realisable tropical oriented matroid with parameters  $(3, 3)$ . In this realisation every cell in the convex hull intersects a tropical line segment between points in  $A$  and points in  $B$ . Note though that there are other realisations of the same tropical oriented matroid where this does not hold. (Imagine shifting the apex of the second tropical hyperplane further to the right until it is no longer possible to draw a line segment from  $A$  to  $B$  through the cell  $(1, 1, 3)$ .)

The following proposition establishes a connection between the combinatorial convex hull and the elimination property.

**Proposition 4.2.** *The types of the cells in a mixed subdivision  $S$  of  $n\Delta^{d-1}$  satisfy the elimination property if and only if  $S_{AB}$  is path-connected (as a subcomplex of  $S$ ) for every  $A, B \in S$ .*

*Proof.* The convex hull  $S_{AB}$  clearly contains each elimination of  $A$  and  $B$ . If  $S_{AB}$  is path-connected then there is a path from  $A$  to  $B$  in  $S_{AB}$ . For any given  $j \in [n]$  this path must contain a cell  $C$  with  $C_j = A_j \cup B_j$ . Then  $C$  works as elimination for  $A$  and  $B$  with respect to  $j$ .

Conversely, assume that  $S$  satisfies the elimination property and fix  $A, B \in S$ . We have to show that there exists a path from  $A$  to  $B$  in  $S_{AB}$ .

Denote  $\text{dist}(A, B) := \{i \mid A_i \not\subseteq B_i, B_i \not\subseteq A_i\}$ . If  $\#\text{dist}(A, B) = 0$  then  $A \cap B \in S_{AB}$  and we are done. Otherwise choose some position  $i \in \text{dist}(A, B)$  and let  $C$  denote the elimination of  $A$  and  $B$  with respect to  $i$ . Then  $C \in S_{AB}$  and we will now show that  $\#\text{dist}(A, C), \#\text{dist}(B, C) \leq \#\text{dist}(A, B) - 1$ .

Indeed consider  $j \notin \text{dist}(A, B)$ . Then  $j \notin \text{dist}(A, C)$  follows immediately. Moreover,  $i \in \text{dist}(A, B) \setminus \text{dist}(A, C)$ . Thus  $\#\text{dist}(A, C) \leq \#\text{dist}(A, B) - 1$  and similarly for  $\text{dist}(B, C)$ .

The claim then follows by iterating this process.  $\square$

**Corollary 4.3.** *A convex set in a tropical oriented matroid is path-connected.*

*Proof.* Since tropical oriented matroids satisfy the elimination property, Proposition 4.2 implies that the convex hull of any two types is path-connected.  $\square$

## 5 The Topological Representation Theorem

This section comprises the long and winding road towards the Topological Representation Theorem for tropical oriented matroids. Note that a different version (with a different definition of tropical pseudohyperplane arrangements) is contained in [H12b].

We first introduce tropical pseudohyperplanes:

**Definition 5.1** (*Cf.* [H12b, Definition 4.3]). A *tropical pseudohyperplane* is the image of a tropical hyperplane under a PL-homeomorphism of  $\mathbb{TP}^{d-1}$  that fixes the boundary.

By [H12b, Theorem 4.2] the Poincaré dual of a mixed subdivision of  $n\Delta^{d-1}$  is a family of tropical pseudohyperplanes.

### 5.1 Linear and affine pseudohyperplanes

Locally, (*i.e.*, in the parallelepiped cells of their mixed subdivisions) we want tropical pseudohyperplanes to intersect as “ordinary” hyperplanes. We thus introduce arrangements of linear pseudohyperplanes on the basis of arrangements of pseudospheres as defined in Björner, Las Vergnas, Sturmfels, White and Ziegler [BLS+99, Def. 5.1.3].

**Definition 5.2** (*Cf.* [BLS+99, Definition 5.1.3]). A *pseudohyperplane* is a set that is PL-homeomorphic to a linear hyperplane. A finite collection  $\mathcal{A} = (H_e)_{e \in E}$  of pseudohyperplanes is called an *arrangement of pseudohyperplanes* if the following conditions hold:

1.  $H_A := \bigcap_{e \in A} H_e$  is a pseudohyperplane of smaller dimension for all  $A \subseteq E$ .
2. If  $H_A \not\subseteq H_e$  for  $A \subseteq E, e \in E$  and  $H_e^+$  and  $H_e^-$  are the two sides of  $H_e$ , then  $H_A \cap H_e$  is a pseudohyperplane in  $H_A$  with sides  $H_A \cap H_e^+$  and  $H_A \cap H_e^-$ .
3. The intersection of an arbitrary collection of closed sides is a ball.

We now define arrangements of *affine* pseudohyperplanes as a generalisation of the above:

**Definition 5.3.** An *arrangement of affine pseudohyperplanes* is a collection  $\mathcal{A}$  of pseudohyperplanes such that for any  $\mathcal{A}' \subseteq \mathcal{A}$  either  $\bigcap_{a \in \mathcal{A}'} H_a = \emptyset$  or  $\mathcal{A}'$  is an arrangement of linear pseudohyperplanes as defined in Definition 5.2.

**Proposition 5.4.** *The intersection of any number of closed pseudohalfspaces in an arrangement of affine pseudohyperplanes in  $\mathbb{R}^d$  is path-connected.*

*Proof.* Let  $H_i, 1 \leq i \leq n$  be affine pseudohyperplanes in  $\mathbb{R}^d$  and denote by  $H_i^+$  the corresponding closed pseudohalfspaces.

The proof will be done by induction on the number  $n$  of pseudohyperplanes, the case where  $n = 1$  being trivially clear.

Assume  $n \geq 2$  and choose two points  $x, y$  in  $\bigcap_{i=1}^n H_i^+$ . By induction there is a path  $p$  from  $x$  to  $y$  in  $\bigcap_{i=1}^{n-1} H_i^+$ . Assume without loss of generality that whenever  $p$  intersects  $H_n$ , it crosses it. (Otherwise we can modify  $p$  to achieve this.)

If  $H_n$  does not intersect  $p$ , we are done since then  $p \subseteq \bigcap_{i=1}^n H_i^+$ . If  $H_n$  intersects  $p$ , then it does so an even number of times. (Walking along  $p$ , at each intersection point we switch between  $H_n^+$  and  $H_n^-$ .) Let  $q, q'$  be the first two intersection points.

We have to find a path  $p'$  from  $q$  to  $q'$  in  $\bigcap_{i=1}^{n-1} H_i^+$ . We will prove the existence of  $p'$  by induction on the dimension  $d$ .

We start with the case  $d = 2$ . *I.e.*, the  $H_i$  are 1-dimensional. Define  $p'$  to be the segment of  $H_n$  between  $q$  and  $q'$ . Then  $p'$  lies in  $\bigcap_{i=1}^{n-1} H_i^+$ . Indeed, assume that there is  $1 \leq i \leq n-1$  such that  $H_i \cap p' \neq \emptyset$ . Then  $p'$  and the segment of  $p$  between  $q$  and  $q'$  form a PL-1-sphere  $S$ . Since the intersection of  $H_i$  and  $H_n$  is a crossing,  $H_i$  enters the interior of  $S$  and hence has to intersect  $S$  a second time by the Jordan curve theorem. Since  $H_i \cap p = \emptyset$ , there is a second intersection point of  $H_i$  with  $p'$ . This is a contradiction.

See Figure 6 for an illustration.

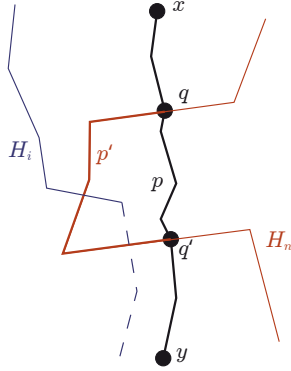


Figure 6: The 2-dimensional situation in the proof of Proposition 5.4.

Now assume  $d \geq 3$ .

Denote  $H'_i := H_i \cap H_n$  and  $(H'_i)^+ := H_i^+ \cap H_n$  for  $1 \leq i \leq n-1$ . Then  $\{H'_i\}$  is an arrangement of affine pseudohyperplanes in  $H_n \stackrel{\text{PL}}{\simeq} \mathbb{R}^{d-1}$  and  $q, q' \in \bigcap_{i=1}^{n-1} (H'_i)^+$ . By induction this set is path-connected.

Hence there is a path  $p'$  from  $q$  to  $q'$  in  $\bigcap_{i=1}^{n-1} (H'_i)^+ \subset \bigcap_{i=1}^n H_i^+$ . Replace the segment of  $p$  between  $q$  and  $q'$  by  $p'$  and continue in the same way for the other intersection points.

Thus, we constructed a path from  $x$  to  $y$  in  $\bigcap_{i=1}^n H_i^+$ . Since  $x$  and  $y$  were arbitrary, this proves that  $\bigcap_{i=1}^n H_i^+$  is path-connected.  $\square$

## 5.2 Arrangements of tropical pseudohyperplanes II

We now define arrangements of tropical pseudohyperplanes. Note that a second definition of tropical pseudohyperplane arrangements is given in [H12b, Definition 4.3]. We will eventually see that both definitions are equivalent.

Let  $H$  be a  $(d-2)$ -dimensional tropical pseudohyperplane in  $\mathbb{T}^{d-1}$ . Then  $H$  divides  $\mathbb{T}^{d-1} \setminus H$  into  $d$  connected components  $S_1, \dots, S_d$ , the *open sectors* of  $H$ . The closure of any union  $\bigcup_{i \in I} S_i$  with  $\emptyset \neq I \subset [d]$  will be called a *(tropical) pseudohalfspace* of  $H$ . We denote by

$$H_I := \overline{\bigcup_{i \in I} S_i} = \overline{\bigcup_{i \notin I} S_i}$$

the boundary of the pseudohalfspace and by

$$H_I^+ := \bigcup_{i \in I} S_i \setminus H_I, \quad \text{respectively} \quad H_I^- := \bigcup_{i \notin I} S_i \setminus H_I$$

the two open pseudohalfspaces. Note that the boundary  $H_I$  of a tropical pseudohalfspace is a (linear) pseudohyperplane with sides  $H_I^+$  and  $H_I^-$ .

An  $(n, d)$ -halfspace system is a tuple  $\mathcal{I} = (I_1, \dots, I_n)$  with  $\emptyset \neq I_i \subset [d]$  for each  $1 \leq i \leq n$ . Given a halfspace system  $\mathcal{I}$  and a collection  $\mathcal{A} = (H_i)_{i \in [n]}$  of  $n$  tropical pseudohyperplanes we write

$$\mathcal{A}_{\mathcal{I}} := \{H_{i, I_i} \mid 1 \leq i \leq n\}.$$

The following definition of tropical pseudohyperplane arrangements is motivated by Propositions 4.2 and 5.4, *i.e.*, by the fact that we want to show that the combinatorial convex of hull of two types is path-connected and know that the intersection of affine pseudohalfspaces is so.

**Definition 5.5.** An *arrangement of tropical pseudohyperplanes* (in weakly general position) is a collection  $\mathcal{A}$  of  $n$  tropical pseudohyperplanes in  $\mathbb{T}^{d-1}$  such that  $\mathcal{A}_{\mathcal{I}}$  forms an arrangement of affine pseudohyperplanes as defined in Definition 5.3 for every  $(n, d)$ -halfspace system  $\mathcal{I}$ .

See Figure 7 for examples of arrangements of tropical pseudohyperplanes in  $\mathbb{T}^3$ .

For a set  $I \subseteq [n]$  we denote its complement by  $\bar{I} := [n] \setminus I$ . For a tropical pseudohyperplane  $H$  and a halfspace  $\emptyset \neq I \subset [d]$  we define

$$\begin{aligned} \mathcal{T}_I : \mathcal{C}(H) &\rightarrow \{+, -, 0\} \\ C &\mapsto \begin{cases} + & \text{if } C \subseteq I, \\ - & \text{if } C \subseteq \bar{I} = [d] \setminus I, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now let  $\mathcal{A}$  be a tropical pseudohyperplane arrangement and  $\mathcal{C}(\mathcal{A})$  the induced cell decomposition of  $\mathbb{T}^{d-1}$ . For  $\mathcal{A}' \subseteq \mathcal{A}$  we define

$$\begin{aligned} \mathcal{T}_{\mathcal{I}} : \mathcal{C}(\mathcal{A}') &\rightarrow \{+, -, 0\}^{\mathcal{A}'} : \\ C &\mapsto (\mathcal{T}_{I_i}(C_i))_i \end{aligned}$$

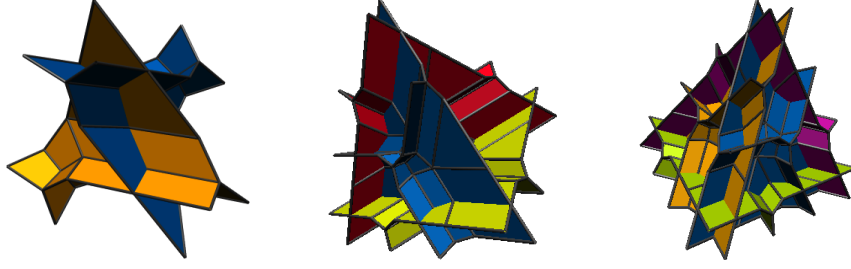


Figure 7: Arrangements of 2-dimensional tropical pseudohyperplanes that are dual to mixed subdivisions of dilated simplices. The arrangement on the right is non-realizable. The pictures were produced with the `polymake` extension `tropmat` [H12c].

and

$$\mathcal{L}(\mathcal{A}', \mathcal{I}) := \{\mathcal{T}_{\mathcal{I}}(C) \mid C \in \mathcal{C}(\mathcal{A}')\}.$$

**Proposition 5.6.** *Let  $M$  be a tropical oriented matroid in general position and  $S$  its corresponding fine mixed subdivision of  $n\Delta^{d-1}$ . Moreover, fix a halfspace system  $\mathcal{I}$ . Then*

- either  $0 \notin \mathcal{L}(\mathcal{A}', \mathcal{I})$  or
- $(\mathcal{L}(\mathcal{A}', \mathcal{I}), \mathcal{A}')$  is an oriented matroid with covectors  $\{0, +, -\}^{\#\mathcal{A}'}$ .

*Proof.* Let  $\mathcal{L} := \mathcal{L}(\mathcal{A}', \mathcal{I})$  and assume  $0 \in \mathcal{L}$ . (Otherwise there is nothing to prove.)

We show that  $\mathcal{L} = \{+, -, 0\}^{\mathcal{A}'}$ . Choose  $A \in \mathcal{T}_{\mathcal{I}}^{-1}(0)$ . Then one can for any  $X \in \{+, -, 0\}^{\mathcal{A}'}$  construct a type  $B \subseteq A$  with  $\mathcal{T}_{\mathcal{I}}(B) = X$ . So define  $B$  by

$$B_i = \begin{cases} A_i & \text{if } X_i = 0 \\ A_i \cap \mathcal{I}_i & \text{if } X_i = + \\ A_i \cap \mathcal{I}_i & \text{if } X_i = -. \end{cases}$$

Then  $B \subseteq A$  and since  $M$  is in general position  $B$  is a refinement of  $A$ . Moreover,  $\mathcal{T}_{\mathcal{I}}(B) = X$ .  $\square$

If  $J_i \subseteq [d]$  for each  $i \in [n]$  and the  $J_i$  are pairwise disjoint then we denote by  $J_1 \cup \dots \cup J_n$  the *partition* of  $\bigcup_i J_i$  into the  $J_i$ .

Now let  $\mathcal{J} = (J_1, \dots, J_n)$  be an  $n$ -tuple of partitions of  $[d]$ . I.e.,  $J_i = (J_{i,1} \cup \dots \cup J_{i,k_i})$  is a partition of  $[d]$  for each  $i \in [n]$ . For a tropical oriented matroid  $M$  denote by

$$M_{\mathcal{J}} := \{A \in M \mid A_i \cap J_{i,k} \neq \emptyset, i \in [n], k \in [k_i]\}$$

the set containing all types in  $M$  all of whose entries intersect each element in the according partition. As before, let  $\mathcal{I} = (I_1, \dots, I_n)$  be an  $n$ -tuple of non-empty subsets of  $[d]$ . Then we denote

$$M_{\mathcal{I}} := \{A \in M \mid A_i \subseteq I_i, i \in [n]\}.$$

Finally, we define

$$M(\mathcal{I}, \mathcal{J}) := M_{\mathcal{I}} \cap M_{\mathcal{J}}.$$

See Figure 8 for an illustration of  $M(\mathcal{I}, \mathcal{J})$ .

**Lemma 5.7.** *Let  $M$  be a tropical oriented matroid in general position. Then  $M(\mathcal{I}, \mathcal{J})$ , if non-empty, is connected and pure of dimension  $d + n - 1 - \sum \#J_i$ .*

*Proof.* We first show that  $M(\mathcal{I}, \mathcal{J})$  is connected: Let  $A, B \in M(\mathcal{I}, \mathcal{J})$ . Then  $A_i, B_i \subseteq I_i$  and  $A_i \cap J_{i,k}, B_i \cap J_{i,k} \neq \emptyset$  for each  $i \in [n]$  and  $k \in [k_i]$ . But this implies  $A_i \cup B_i \subseteq I_i$  and  $(A_i \cup B_i) \cap J_{i,k} \neq \emptyset$ . Hence  $M(\mathcal{I}, \mathcal{J})$  is convex in the sense of Definition 4.1 and thus connected by Proposition 4.2.

It remains to show that  $M(\mathcal{I}, \mathcal{J})$  is pure of the correct dimension. Let  $A \in M(\mathcal{I}, \mathcal{J})$ . Since  $A_i \cap J_{i,k} \neq \emptyset$ , it follows that  $\#A_i \geq \#J_i$  for each  $i$ . Hence  $\dim A \leq d + n - 1 - \sum \#J_i$ .

Since  $M$  is in general position we can construct a type  $B \subseteq A$  with  $\#B_i \cap J_{i,k} = 1$  for every  $i, k$  by deleting sufficiently many elements from the entries of  $A$ . Then  $\dim B = d - 1 - \sum (\#J_i - 1) = d + n - 1 - \sum \#J_i$ . Since  $A$  was arbitrary this shows that any type in  $M(\mathcal{I}, \mathcal{J})$  is contained in one of dimension  $d + n - 1 - \sum \#J_i$ .  $\square$

For a cell complex  $\mathcal{C}$  we denote by  $\overline{\mathcal{C}}$  its *closure*, i.e.,  $\overline{\mathcal{C}}$  consists of all cells of  $\mathcal{C}$  and their faces.

**Lemma 5.8.** *Let  $M, \mathcal{I}, \mathcal{J}$  as before. Then  $\overline{M(\mathcal{I}, \mathcal{J})}$  is a PL-manifold with boundary.*

*Proof.* Denote  $\mathcal{M} := \overline{M(\mathcal{I}, \mathcal{J})}$  and  $\mathcal{M}' := M_{\mathcal{J}}$ . Choose a cell  $T \in \mathcal{M}$ . We first investigate the link  $\text{lk}_{\mathcal{M}'} T$ . The cells in  $\text{lk}_{\mathcal{M}'} T$  correspond to the cells in the star  $\text{st}_{\mathcal{M}'} T = \{C \in \mathcal{M}' \mid C \subseteq T\}$  and hence to certain refinements of  $T$ . First assume that  $n = 1 = k_1$ , i.e.,  $\mathcal{J} = (J_1 = (J_{11}))$ . Then the cells in  $\text{st}_{\mathcal{M}'} T$  are in bijection with the proper subsets of  $J_{11} \cap T_1$  ordered by reverse inclusion. Hence  $\text{lk}_{\mathcal{M}'} T$  is the boundary of a simplex of dimension  $\#(J_{11} \cap T_1) - 1$  (whose facets are labelled by  $J_{11} \cap T_1$ ).

Since  $M$  is in general position we can consider the  $J_{ik}$  (for  $i \in [n], k \in [k_i]$ ) independently. I.e., in general,  $\text{lk}_{\mathcal{M}'} T$  is the boundary of a product of simplices (one for each  $J_{ik}$ ) and hence a PL-sphere. Denote this sphere by  $\mathcal{S}(T)$ . See Figures 8(b) and (c) for an example.

If in each position  $i$  there is some  $J_{ik}$  with  $J_{ik} \cap T_i \subseteq I_i$  then  $T$  is contained in the interior of  $\mathcal{M}$  and  $\text{lk}_{\mathcal{M}} T = \mathcal{S}(T)$ . Otherwise denote by  $\mathcal{B}(T)$  the set of all faces of  $\mathcal{S}(T)$  that do *not* belong to  $\text{lk}_{\mathcal{M}} T$ . Then define  $\mathcal{J}'$  by replacing each  $J_i$  in  $\mathcal{J}$  by  $(I_i \cup (J_{i1} \cap \overline{I_i}) \cup \dots \cup (J_{ik_i} \cap \overline{I_i}))$ . Then  $\overline{\mathcal{B}(T)} \cap \text{lk}_{\mathcal{M}} T = M_{\mathcal{J}'}$  is a PL-sphere in  $\mathcal{S}(T)$  with sides  $\mathcal{B}(T)$  and  $\text{lk}_{\mathcal{M}} T$ . By [BLS+99, Lemma 5.1.1] this implies that  $\text{lk}_{\mathcal{M}} T$  is a PL-ball.

It remains to show that  $\mathcal{M}$  has a boundary. If there is a cell  $T$  whose link is a ball we are done. Otherwise – unless  $\mathcal{M}$  consists of a single point – we can always construct a cell in  $\mathcal{M}$  whose dual (in the mixed subdivision corresponding to  $M$ ) is contained in the boundary of  $n\Delta^{d-1}$ . (Note that we have to view  $\mathcal{M}$  as a manifold in  $\mathbb{TP}^{d-1}$  for it to be compact.) Indeed the cells in the boundary of  $n\Delta^{d-1}$  are characterised by the fact that their types are unbounded, i.e., there is some  $i \in [n]$  not contained in any position of the type. The only situation, however, when  $i \in [n]$  is contained in any cell in  $\mathcal{M}$  is

when any  $J_i$  contains a singleton  $\{i\}$ . If this holds for every  $i$  then  $\mathcal{M}$  consists of one point only.  $\square$

### 5.3 Constructibility

In the proof of the Topological Representation Theorem for classical oriented matroids given in [BLS+99], the *shellability* of certain complexes plays a crucial role. In particular, the fact that a shellable PL-manifold is either a ball or a sphere is used in order to show that the subcomplexes which one would like to be pseudospheres actually are pseudospheres.

In the proof of the tropical analogue, we are going to apply a related but weaker notion, namely that of constructibility.

The notion of constructibility of a polytopal complex goes back to Hochster [Hoc72].

**Definition 5.9.** A polyhedral  $d$ -complex  $C$  is *constructible* if

- $C$  consists of only one cell or
- $C = C_1 \cup C_2$ , where  $C_1, C_2$  are  $d$ -dimensional constructible complexes and  $C_1 \cap C_2$  is a  $(d - 1)$ -dimensional constructible complex.

**Proposition 5.10.** *Let  $M, \mathcal{I}, \mathcal{J}$  as before. Then  $M(\mathcal{I}, \mathcal{J})$  is constructible.*

*Proof.* We are done if  $M(\mathcal{I}, \mathcal{J})$  consists of one (maximal) cell only. Otherwise there are two maximal cells  $A$  and  $B$ . By Lemma 5.7 above (and the fact that  $A, B$  are maximal) we then have  $\#A_i = \#B_i$  and  $\#A_i \cap J_{i,j} = \#B_i \cap J_{i,j} = 1$  for every  $i$  and  $j$ .

There is some position  $k$  where  $A$  and  $B$  differ. Moreover, there is some  $\ell$  with  $J_{k,\ell} \cap A_k \neq J_{k,\ell} \cap B_k$ . Let  $a \in J_{k,\ell} \cap A_k, b \in J_{k,\ell} \cap B_k$ . (Note that  $a$  and  $b$  are unique.)

Now form  $\mathcal{J}_0$  by splitting  $J_{k,\ell}$  so that  $a$  and  $b$  are in different sets. Moreover, form  $\mathcal{I}_1, \mathcal{I}_2$  by removing  $a$ , respectively  $b$  from  $I_k$ . Then  $\overline{M(\mathcal{I}, \mathcal{J})} = \overline{M(\mathcal{I}_1, \mathcal{J})} \cup \overline{M(\mathcal{I}_2, \mathcal{J})}$  and  $\overline{M(\mathcal{I}_1, \mathcal{J})} \cap \overline{M(\mathcal{I}_2, \mathcal{J})} = \overline{M(\mathcal{I}, \mathcal{J}_0)}$ . Moreover,  $A \in \overline{M(\mathcal{I}_1, \mathcal{J})}, B \in \overline{M(\mathcal{I}_2, \mathcal{J})}$ . By the above lemma,  $M(\mathcal{I}_1, \mathcal{J}), M(\mathcal{I}_2, \mathcal{J}), M(\mathcal{I}, \mathcal{J}_0)$  are connected and pure and of the right dimensions. By induction these three sets are constructible and hence  $\overline{M(\mathcal{I}, \mathcal{J})}$  is constructible.  $\square$

See Figure 8 for an illustration.

The above lemmas together with a theorem by Zeeman ([Zee63], “A constructible manifold with a boundary is a ball.”) yield:

**Proposition 5.11.** *Let  $M$  be a tropical oriented matroid in general position. Then  $M(\mathcal{I}, \mathcal{J})$  is a PL-ball.*

*Proof.*  $\overline{M(\mathcal{I}, \mathcal{J})}$  is constructible and pure of dimension  $d + n - 1 - \sum \#J_i$  by Lemma 5.7 and Proposition 5.10.

By Lemma 5.8,  $\overline{M(\mathcal{I}, \mathcal{J})}$  is a PL-manifold with boundary and hence a PL-ball by Zeeman’s theorem.  $\square$



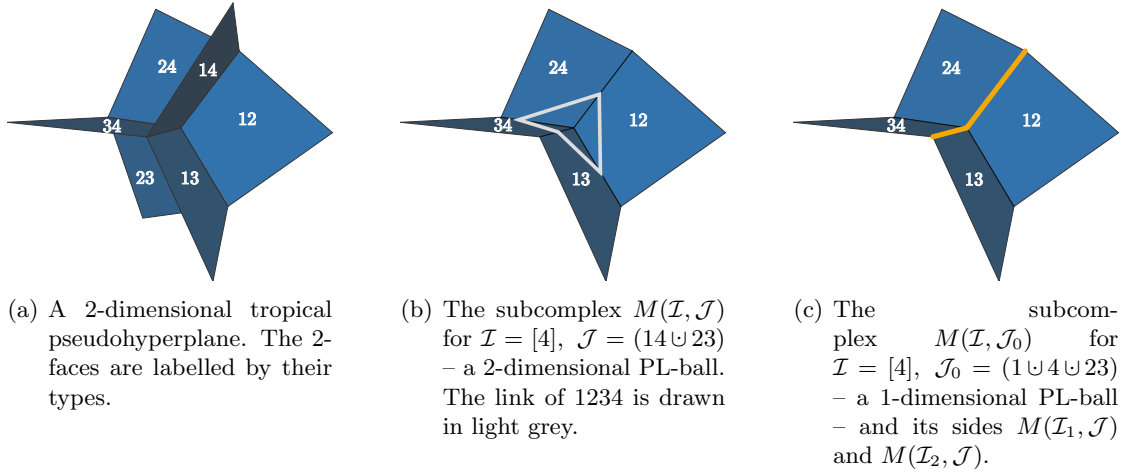


Figure 8: Assume in the proof of Proposition 5.10 we have  $n = 1, d = 4$ , i.e., we are dealing with a 2-dimensional tropical pseudohyperplane as depicted in Figure (a). Moreover, assume we have  $M(\mathcal{I}, \mathcal{J})$  with  $\mathcal{I} = [4]$ ,  $\mathcal{J} = (14 \cup 23)$ . The complex  $M(\mathcal{I}, \mathcal{J})$  is depicted in Figure (b). Now let  $A = 13, B = 24$ . As in the proof we see that  $\#A_1 = \#B_1$  and  $\#A_1 \cap J_{1i} = \#B_1 \cap J_{1j} = 1$  for every  $i$  and  $j$ . We have  $k = 1$  and we may choose  $\ell = 1$ . Then we get  $a = 1, b = 4$  as the unique elements in  $A_1 \cap J_{11}, B_1 \cap J_{11}$ . We form  $\mathcal{J}_0 = (1 \cup 4 \cup 23)$  by splitting  $J_{k\ell} = 14$ . Moreover, we set  $\mathcal{I}_1 = 234$  and  $\mathcal{I}_2 = 123$ . This situation is depicted in Figure (c).

**Corollary 5.12.** *Let  $M$  be a tropical oriented matroid in general position and  $S$  its corresponding fine mixed subdivision of  $n\Delta^{d-1}$ . Moreover, choose a halfspace system  $\mathcal{I}$  and  $X \in \{+, -, 0\}^n$ . Then  $\mathcal{T}_{\mathcal{I}}^{-1}(X)$  is a PL-ball of dimension  $d - 1 - \#z(X)$ , where  $z(X)$  denotes the zero set of  $X$ .*

*Proof.* Define  $\mathcal{I}' = (I'_1, \dots, I'_n)$  by

$$I'_i := \begin{cases} I_i & \text{if } X_i = +, \\ \overline{I_i} & \text{if } X_i = -, \\ [d] & \text{if } X_i = 0 \end{cases}$$

and  $\mathcal{J} = (J_1, \dots, J_n)$  by

$$J_i := \begin{cases} [d] & \text{if } X_i \in \{+, -\}, \\ I_i \cup \overline{I_i} & \text{if } X_i = 0. \end{cases}$$

Then  $\mathcal{T}_{\mathcal{I}}^{-1}(X) = M(\mathcal{I}', \mathcal{J})$  and hence the claim follows from Proposition 5.11.  $\square$

We are now ready to prove the following version of the Topological Representation Theorem for tropical oriented matroids:

**Theorem 5.13.** *Every tropical oriented matroid in general position can be realised by an arrangement of tropical pseudohyperplanes as in Definition 5.5.*

*Proof.* Let  $M$  be a tropical oriented matroid in general position,  $S$  the fine mixed subdivision of  $n\Delta^{d-1}$  corresponding to  $M$  and  $\mathcal{A}$  the family of tropical pseudohyperplanes induced by  $S$ . We have to show that  $\mathcal{A}'_{\mathcal{I}}$  is an arrangement of affine pseudohyperplanes for each  $\mathcal{A}' \subseteq \mathcal{A}$  and halfspace system  $\mathcal{I} = (I_1, \dots, I_n)$ .

So assume that  $\bigcap \mathcal{A}'_{\mathcal{I}} \neq \emptyset$ , i.e.,  $0 \in \mathcal{L}(\mathcal{A}', \mathcal{I})$ . Hence by Proposition 5.6  $(\mathcal{L}(\mathcal{A}', \mathcal{I}), \mathcal{A}')$  is an oriented matroid given by its covectors.

We have to show that  $\mathcal{A}'_{\mathcal{I}}$  satisfies the axioms in Definition 5.2.

1. Let  $A \subseteq \mathcal{A}'_{\mathcal{I}}$ . We have to show that  $H_A := \bigcap_{a \in A} H_a$  is a PL-ball. So let  $\mathcal{I}' = (I'_1, \dots, I'_n)$  with  $I'_i = [d]$  for each  $i$  and  $\mathcal{J} = (J_1, \dots, J_n)$  with

$$J_i = \begin{cases} I_i \cup \overline{I_i} & \text{if } i \in A, \\ [d] & \text{otherwise.} \end{cases}$$

Then  $H_A = M(\mathcal{I}', \mathcal{J})$ .

2. Assume  $e \notin A$ . Then  $H_A \not\subseteq H_e$ . We have to show that  $H_A \cap H_e$  is a pseudohyperplane in  $H_A$  with sides  $H_A \cap H_e^+$  and  $H_A \cap H_e^-$ .

To this end let  $\mathcal{I}', \mathcal{J}$  as before. Moreover, define  $\mathcal{I}'_1, \mathcal{I}'_2$  by

$$I'_{1,i} = \begin{cases} I_i & \text{if } i = e, \\ [d] & \text{otherwise,} \end{cases}$$

$$I'_{2,i} = \begin{cases} \overline{I_i} & \text{if } i = e, \\ [d] & \text{otherwise} \end{cases}$$

and  $\mathcal{J}_0$  by

$$J_{0,i} = \begin{cases} I_i \cup \overline{I_e} & \text{if } i = e, \\ J_i & \text{otherwise.} \end{cases}$$

Then  $H_A \cap H_e = M(\mathcal{I}', \mathcal{J}_0)$ ,  $H_A \cap H_e^+ = M(\mathcal{I}'_1, \mathcal{J})$  and  $H_A \cap H_e^- = M(\mathcal{I}'_2, \mathcal{J})$ . Since  $\bigcap \mathcal{A}'_{\mathcal{I}} \neq \emptyset$ , each of  $H_A \cap H_e$ ,  $H_A \cap H_e^+$  and  $H_A \cap H_e^-$  is non-empty by Proposition 5.6.

Hence  $H_A \cap H_e, H_A \cap H_e^+$  and  $H_A \cap H_e^-$  are PL-balls of the correct dimensions. Moreover,  $\overline{H_A \cap H_e^+} \cap \overline{H_A \cap H_e^-} = H_A \cap H_e$  and hence  $H_A \cap H_e^+$  and  $H_A \cap H_e^-$  are the sides of  $H_A \cap H_e$ .

3. We have to show that the intersection of an arbitrary collection of closed sides is a PL-ball. This follows directly from Corollary 5.12.  $\square$

## 6 The elimination property

This section is about the all important elimination property. Recall that by Oh and Yoo [OY11, Proposition 4.12] the elimination property holds for fine mixed subdivisions of  $n\Delta^{d-1}$ . In this section we apply the Topological Representation Theorem 5.13 to extend this to all mixed subdivisions of  $n\Delta^{d-1}$ .

### 6.1 Blowing up hyperplanes in a mixed subdivision

Let  $S$  be a fine mixed subdivision of  $n\Delta^{d-1}$  and fix  $i \in [n]$ . The following construction is an inverse of the deletion operation and yields a mixed subdivision of  $N\Delta^{d-1}$  ( $N > n$ ) by “blowing up” one tropical pseudohyperplane in the dual arrangement.

We have to fix some notation: Let  $S$  be a fine mixed subdivision of  $n\Delta^{d-1}$ . For  $\emptyset \neq I \subset [n]$  we denote by  $S|_I$  the mixed subdivision of  $n\Delta^{\#I-1}$  induced by  $S$  on the  $I$ -face of  $n\Delta^{d-1}$ . I.e.,  $S|_I$  is the contraction  $S_{/\overline{I}}$  of  $S$  with the complement of  $I$ .

**Definition 6.1.** Let  $S, S'$  be fine mixed subdivisions of  $n\Delta^{d-1}$ , respectively  $n'\Delta^{d-1}$ . Let  $C \in S$  be a cell. Then the *blow-up* of  $C$  with respect to  $S'$  at position  $i$  is the set of  $(n + n' - 1, d)$ -types

$$C \vee_i S' := \{(C \vee_i, X) \mid X \in S'|_{C_i}\}.$$

That is, we subdivide the  $C_i$ -face of  $C$  as  $S'|_{C_i}$ .

Moreover, the *blow-up* of  $S$  with respect to  $S'$  at position  $i$  is

$$S \vee_i S' := \bigcup_{C \in S} C \vee_i S'.$$

See Figure 9 for an example.

The following lemma follows easily:

**Lemma 6.2.** *The types in the blow-up  $S \vee_i S'$  yield a fine mixed subdivision of  $N\Delta^{d-1}$  with  $N := n + n' - 1$ .*

*Proof.* It is clear that each type corresponds to a Minkowski cell inside  $N\Delta^{d-1}$  and that the cells cover  $N\Delta^{d-1}$ . It remains to show the intersection property.

Let  $A = A_S \vee_i A_{S'}$ ,  $B = B_S \vee_i B_{S'}$  be two cells in  $S \vee_i S'$ . We have to show that  $A$  and  $B$  are comparable. Since  $S$  is a mixed subdivision,  $A_S$  and  $B_S$  are comparable, i.e.,  $\mathbb{G}_{A_S, B_S}$  is acyclic. The same holds for  $\mathbb{G}_{A_{S'}, B_{S'}}$ .

Now consider the comparability graph  $\mathbb{G}_{A, B}$ . This has the same vertex set  $[d]$  and all edges from  $\mathbb{G}_{A_S, B_S}$  accounting for positions different from  $i$  and all edges from  $\mathbb{G}_{A_{S'}, B_{S'}}$ .

For position  $i$ , the graph  $\mathbb{G}_{A_S, B_S}$  contains one edge (directed or undirected) between  $a$  and  $b$  for every  $a \in A_{S, i}$ ,  $b \in B_{S, i}$ ,  $a \neq b$ . The edge set of  $\mathbb{G}_{A_{S'}, B_{S'}}$  is a subset of the set of these edges. An undirected edge in  $\mathbb{G}_{A_S, B_S}$  might, however, correspond to a directed one in  $\mathbb{G}_{A_{S'}, B_{S'}}$ . Since  $S'$  is a mixed subdivision, the graph  $\mathbb{G}_{A_{S'}, B_{S'}}$  is acyclic.

Hence it remains to exclude that an undirected cycle in  $\mathbb{G}_{A_S, B_S}$  becomes a directed one in  $\mathbb{G}_{A, B}$ . But since  $S$  is fine, for any undirected edge in  $\mathbb{G}_{A_S, B_S}$  there is a unique position accounting for this edge. Moreover, any undirected cycle in  $\mathbb{G}_{A_S, B_S}$  would yield a cycle in the type graphs of  $A_S$  and  $B_S$  which do not exist since  $S$  is fine.  $\square$

Now fix some permutation  $\pi$  of  $[d]$ . Let  $S_\pi$  be the  $n$ -placing extension of  $\Delta^{d-1}$  with respect to  $\pi$ . Then we define the *blow-up* of the  $i$ -th tropical pseudohyperplane in  $S$  with respect to  $\pi$  by

$$S_{i,\pi} := S \vee_i S_\pi.$$

In the dual setting of an arrangement of tropical pseudohyperplanes this blow-up operation corresponds to adding a slightly shifted copy of the  $i$ -th tropical pseudohyperplane.

It is more difficult to define the blow-up of a tropical pseudohyperplane in a mixed subdivision of  $n\Delta^{d-1}$  which is not fine. Let  $S$  be a mixed subdivision of  $n\Delta^{d-1}$ ,  $i \in [n]$  and  $\pi = (\pi_1, \dots, \pi_d) \in \text{Sym}_d$ . We also denote by  $\bar{\pi} := (\pi_d, \dots, \pi_1)$  the permutation obtained by reversing  $\pi$ .

Then the blow-up of the  $i$ -th tropical pseudohyperplane has the following full-dimensional cells:

- If  $A = (A_1, \dots, A_n)$  is a full-dimensional cell in  $S$  with  $\#A_i = 1$  (i.e.,  $A$  is not contained in the  $i$ -th hyperplane), then  $(A, A_i)$  is a maximal cell in  $S_{i,\pi}$ .
- If  $A = (A_1, \dots, A_n)$  is a full-dimensional cell in  $S$  with  $\#A_i \geq 2$  then  $(A, \{\pi_d\})$  is a maximal cell in  $S_{i,\pi}$ .
- Finally, the maximal cells corresponding to the new hyperplane are constructed as follows: Let again  $S_\pi$  denote the  $n$ -placing extension of  $n\Delta^{d-1}$  with respect to  $\pi$ . Let  $P$  be an ordered partition of  $[d]$  that has  $\bar{\pi}$  as a refinement. (I.e., neighbouring entries of  $\bar{\pi}$  may be combined into one set.) Moreover, let  $A$  be a full-dimensional cell in  $S$  with  $\#A_i \geq 2$ . Define  $B := A|_P$  and let  $C$  be the unique maximal cell in  $S_\pi$  with  $C_1 = B_i$ . Then  $(B, C_2)$  is a maximal cell in  $S_{i,\pi}$ .

## 6.2 Approximation by blow-ups

In this section we prove that tropical pseudohyperplane arrangements as defined in Definition 5.5 satisfy the elimination property and use this to show the same for all mixed subdivisions of  $n\Delta^{d-1}$ .

Since it simplifies the presentation we assume all arrangements of tropical pseudohyperplanes in this section to come from a (fine) mixed subdivision of  $n\Delta^{d-1}$ . I.e., we

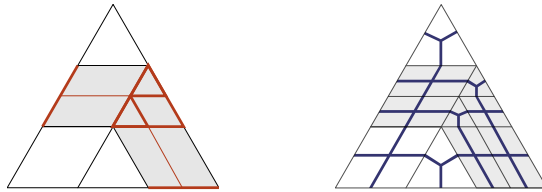


Figure 9: The blow-up of a mixed subdivision of  $3\Delta^2$  with respect to one of  $2\Delta^2$ . The cells in the shaded hyperplane are subdivided according to the subdivision of the small simplex. The according tropical pseudohyperplane arrangement is drawn on the left.

only consider tropical pseudohyperplane arrangements which are dual to a fine mixed subdivision of  $n\Delta^{d-1}$ .

Let  $H$  be a tropical hyperplane with apex 0. Recall that  $H_I$  denotes the boundary of the tropical halfspace separating the points with types in  $I$  from those with types in the complement  $\bar{I}$ . For  $p \in \mathbb{T}^{d-1}$  and  $\emptyset \neq I \subseteq [d]$  denote  $H_{I,p} := H_I - p$ , *i.e.*, we shift the apex of  $H_I$  to  $p$ . For  $\emptyset \neq I \subseteq [d]$  denote by  $T_I$  the set of all points of type  $I$ . Let

$$\mathcal{F} := \{\text{aff } T_I \mid I \in \binom{[d]}{2}\},$$

*i.e.*,  $\mathcal{F}$  is an arrangement of linear hyperplanes in  $\mathbb{T}^{d-1}$ . In fact,  $\mathcal{F}$  is the arrangement of reflection hyperplanes corresponding to the Coxeter group  $A_d$ . The connected components (*sectors*) of  $\mathbb{T}^{d-1} \setminus (\bigcup \mathcal{F})$  correspond one-to-one to the permutations of  $[d]$ : Again, view  $\mathcal{F}$  embedded in the simplex  $\Delta^{d-1}$ . For  $v \in \Delta^{d-1}$  and  $i \in [d]$  denote by  $d_i(v)$  the distance of  $v$  to the  $i$ -th vertex of  $\Delta^{d-1}$ . Then each sector is determined by the permutation of  $[d]$  induced by ordering the  $d_i(v)$  increasingly. The sectors are dual to the vertices of the  $d$ -dimensional permutahedron. See Figure 10 for an illustration.

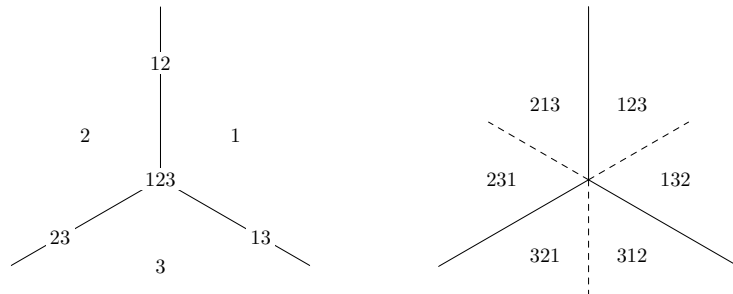


Figure 10: A 2-dimensional tropical hyperplane with its types (on the left) and the corresponding arrangement  $\mathcal{F}$  of hyperplanes (on the right). Moreover, the bijection between the open sectors of  $\mathcal{F}$  and the permutations of  $[3]$  is given.

For  $X \subseteq \{I \mid \emptyset \neq I \subset [d]\}$  we say that  $A \subseteq \mathbb{T}^{d-1}$  *approximates*  $T_X := \bigcup_{I \in X} T_I$  if:

- For each  $I \in X$ , there is  $\varepsilon_I > 0$  such that  $T_I$  is contained in  $A$  except possibly for an  $\varepsilon_I$ -neighbourhood of the (relative) boundary  $\partial T_I$ .
- For each  $I \notin X$  there is  $\varepsilon_I > 0$  such that  $T_I \cap A$  is contained in an  $\varepsilon_I$ -neighbourhood of  $\partial T_I$ .

Intuitively, the set  $A$  is supposed to contain “almost everything” of  $T_I$  if  $I \in X$  and “almost nothing” of  $T_I$  if  $I \notin X$ . Then  $T_X$  is homeomorphic to  $A$ . We will be interested in approximating neighbourhoods for  $X = \{a, b, a \cup b\}$  with  $a, b \subset [d]$ . See Figure 11 for an illustration.

For  $I \subseteq [d]$  and  $p \in \mathbb{T}^{d-1}$  denote by  $\text{App}_{I,p}$  the set of all types that are approximated by  $H_{I,p}^+$ . The following lemma characterises the types in  $\text{App}_{I,p}$ :

**Lemma 6.3.** *Let  $H \subset \mathbb{T}^{d-1}$  be a tropical hyperplane and  $\pi = (\pi_1, \dots, \pi_d)$  the permutation of  $[d]$  corresponding to a point  $p \in \mathbb{T}^{d-1} \setminus (\bigcup \mathcal{F})$ .*

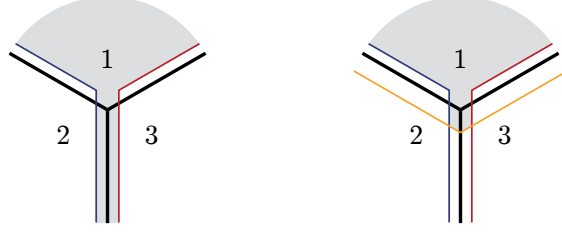


Figure 11: Approximating neighbourhoods corresponding to  $a = 1$  and  $b = 23$  (on the left), respectively  $a = 1$  and  $b = 123$  (on the right).

1. Let  $\emptyset \neq J \subset [d]$ . Then  $T_J \cap H_{i,p}^+ \neq \emptyset$  if and only if each  $j \in J \setminus \{i\}$  comes before  $i$  in  $\pi$ .
2.  $\text{App}_{I,p}$  only depends on the open sector of  $\mathcal{F}$  in which  $p$  lies, hence on the permutation corresponding to  $p$ .
3. Let  $i \in [d]$  and  $J \subseteq [d]$ . Then  $J \in \text{App}_{i,p}$  if and only if  $i \in J$  and  $\{i\} \cup \{q \mid q \text{ comes before } i \text{ in } \pi\} \supseteq J$ .
4.  $\text{App}_{I,p} = \bigcup_{i \in I} \text{App}_{i,p}$ .

*Proof.*

1. We first prove the statement for  $\#J = 1$ . So assume  $J = \{j\}$ . But then it is easy to see that  $H_{i,p}^+$  intersects  $T_j = H_j^+$  if and only if  $j = i$  or  $j$  comes before  $i$  in  $\pi$ .  
The general statement (for  $\#J \geq 2$ ) follows since intersections of tropically convex sets are tropically convex and  $H_{i,p}^+$  is open.
2. This is clear.
3. Assume without loss of generality that  $i = 1$ . We will prove the statement by induction over the length of  $\pi$ , *i.e.*, the minimal number of transpositions needed to write  $\pi$  as a product of transpositions.

It is clear that the only type approximated by  $H_{1,p}^+$  for  $\pi_1 = 1$  is  $\{1\}$ .

Now assume the statement is true for  $\pi = (\pi_1, \dots, \pi_d)$  and apply one transposition  $\tau = (\pi_j, \pi_{j+1})$  with  $\pi_j < \pi_{j+1}$  to obtain  $\pi'$ ; *i.e.*,  $\tau$  swaps two neighbouring entries of  $\pi$ , increasing the length by one. Denote by  $p'$  one point in the  $\pi'$ -sector of  $H$ . In particular, we can always choose  $p'$  such that  $H_{i,p'}^+ \supset H_{i,p}^+$ .

This means we move  $p$  into a neighbouring sector of  $\mathcal{F}$ . There are two cases:

- If both  $\pi_j, \pi_{j+1}$  come before or after 1 in  $\pi$ , the types approximated by  $H_{i,p}^+$  do not change. Indeed, we can decrease the length by one by relabeling the sectors  $\pi_j \leftrightarrow \pi_{j+1}$ .

- Assume  $\pi_j = 1$ . By passing from sector  $\pi$  to the sector  $\pi'$  we cross the hyperplane  $\text{lin } T_{1,\pi_{j+1}}$ . We now show that then  $\text{App}_{i,p'} = \text{App}_{i,p} \cup \{r \cup \{\pi_{j+1}\} \mid r \in \text{App}_{i,p}\}$ .

So let  $r \in \text{App}_{i,p}$  and denote  $r' := r \cup \{\pi_{j+1}\}$ . I.e.,  $T_r$  is approximated by  $H_{i,p}^+$ . But then clearly  $T_r$  is also approximated by  $H_{i,p'}^+ \supset H_{i,p}^+$ . Moreover,  $T_{r'}$  is approximated by  $H_{i,p'}^+$  since it intersects  $H_{i,p'}$  and is contained in the boundary of  $T_r$ . That  $\text{App}_{i,p'}$  is not larger than this follows from (1).

4. This follows from  $H_{I,p}^+ = \bigcup_{i \in I} H_{i,p}^+$ .  $\square$

**Lemma 6.4.** *Let  $H$  be a tropical pseudohyperplane in  $\mathbb{T}^{d-1}$  and  $\emptyset \neq I, J \subset [d]$ . Then we can represent an approximating neighbourhood of  $T_I \cup T_J \cup T_{I \cup J}$  as an intersection of affine pseudohalfspaces.*

*Proof.* It suffices to prove the statement for usual tropical hyperplanes since the PL-homeomorphism taking a tropical hyperplane to a tropical pseudohyperplane also maps our affine pseudohalfspaces in an appropriate way. See Figure 11 for an example.

It suffices to show that for each set  $K \neq I, J, I \cup J$  there are  $L \subset [d]$  and  $p \in \mathbb{T}^{d-1}$  such that  $\text{App}_{L,p}$  contains  $I, J, I \cup J$  but not  $K$ . Then we only need to intersect all of these affine pseudohalfspaces for each  $K \neq I, J, I \cup J$ .

Note that the open sectors of the arrangement  $\mathcal{F}$  of linear hyperplanes (and hence the points  $p \in \mathbb{T}^{d-1} \setminus (\bigcup \mathcal{F})$ ) correspond to permutations in  $\pi \in \text{Sym}_d$ . See again Figure 10.

- First assume that there is  $x \in K \setminus (I \cup J)$ . Then we can choose  $\pi$  to end in  $x$  to make sure  $x$  will never occur in any element of  $\text{App}_{L,p}$ . In detail, choose  $i \in I, j \in J$ . Let  $L = \{i, j\}$  and let  $p$  be such that  $i' \leq i$  for each  $i' \in I$  and  $j' \leq j$  for each  $j' \in J$  and  $x > y$  for each  $y \neq x$ .
- If  $K \subseteq I \cap J$ , choose  $i \in I \setminus K, j \in J \setminus K$ , which exist since  $K \neq I, J$ . Let  $L = \{i, j\}$  and choose  $\pi$  in such a way that the elements of  $I - \{i\}$  and  $J - \{j\}$  come first. Then any element of  $\text{App}_{L,p}$  contains either  $i$  or  $j$ . Thus,  $K \notin \text{App}_{L,p}$  and it is easy to check that  $I, J, I \cup J \in \text{App}_{L,p}$ .
- Otherwise there is  $i \in (I \cup J) \setminus K$ . Let  $L = \{i\}$  and let  $\pi$  begin with the elements of  $(I \cup J) - \{i\}$ . Then every element of  $\text{App}_{L,p}$  contains  $i$ . Hence  $K \notin \text{App}_{L,p}$ . Again, it is easy to see that  $I, J, I \cup J \in \text{App}_{L,p}$ .  $\square$

See Figure 13 for an example.

**Lemma 6.5.** *Let  $H$  be a tropical pseudohyperplane with apex 0 in  $\mathbb{T}^{d-1}$ . For each  $(I, \pi)$  with  $\emptyset \neq I \subset [d]$  and  $\pi \in \text{Sym}_d$  fix one point  $p_{I\pi}$  in the  $\pi$ -sector of  $H$  in such a way that the arrangement of tropical hyperplanes with apices in  $\{0\} \cup \{p_{I\pi}\}$  is in general position. Then*

$$\mathcal{H} := \{H_{I,p_{I\pi}} \mid \emptyset \neq I \subset [d], \pi \in \text{Sym}_d\}$$

*is an arrangement of affine pseudohyperplanes.*

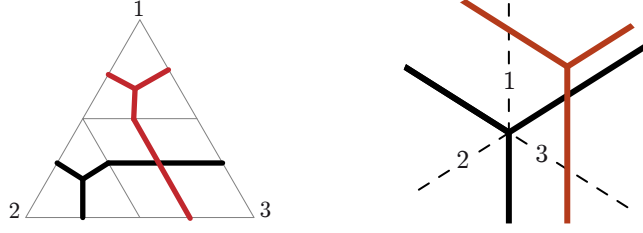


Figure 12: The blow-up of the black tropical pseudohyperplane with respect to  $\pi = (2, 3, 1)$  yields a new tropical pseudohyperplane with apex in the  $(1, 3, 2)$ -sector of the first tropical pseudohyperplane.

*Proof.* This follows by applying the Topological Representation Theorem 5.13 to realisable tropical oriented matroids.  $\square$

We can extend the above construction to tropical *pseudohyperplanes* as follows: Let  $H$  be a tropical pseudohyperplane. Then  $H$  is the image of a tropical hyperplane  $H'$  under a PL-homeomorphism  $\phi$  of  $\mathbb{T}^{d-1}$ . Then we define  $H_{I,p} := \phi(H'_{I,p})$ .

Note that by continuity of  $\phi$  and the fact that  $\phi$  fixes the boundary of  $\mathbb{T}^{d-1}$  we can always choose the point  $p$  so that  $H_{I,p}$  lies very close to  $H_I$ . Now consider an arrangement  $\mathcal{A} = (H_i)_{i \in [n]}$  of tropical pseudohyperplanes. We can do the above construction for each of them individually.

If  $H$  is a tropical pseudohyperplane in such an arrangement, then we can consider  $H_{I,p}$  as  $H'_I$  for the new hyperplane  $H'$  that arises by blowing up  $H$  with respect to the permutation  $p$ . The following is immediate:

**Lemma 6.6.** *Let  $S = \triangle^{d-1}$  be the mixed subdivision dual to a tropical hyperplane  $H$  and fix  $\pi \in \text{Sym}_d$ . Then the blow-up of  $H$  with respect to  $\pi$  corresponds to adding a second tropical hyperplane with apex in the  $\bar{\pi}$ -sector of  $H$ .*

See Figure 12 for an illustration.

We can use blow-ups to construct an affine pseudohyperplane arrangement  $\mathcal{H}$  for a given tropical pseudohyperplane  $H$ . For each  $(I, \pi)$  with  $\pi \in \text{Sym}_d$  and  $\emptyset \neq I \subset [d]$  perform one blow-up of  $H$  with respect to  $\bar{\pi}$  and denote the tropical pseudohyperplane emerging from this blow-up by  $H^{I,\pi}$ .

We then obtain  $(2^d - 2)d!$  new tropical hyperplanes (one for each  $(I, \pi)$  and hence a mixed subdivision of  $((2^d - 2)d! + 1)\triangle^{d-1}$ . With this we can, in the dual arrangement of tropical pseudohyperplanes, define  $\mathcal{H} = H^{I,\pi}$ .

See Figure 13 for an illustration.

**Theorem 6.7.** *The types in a tropical pseudohyperplane arrangement as in Definition 5.5 satisfy the elimination axiom of a tropical oriented matroid.*

*Proof.* Let  $A, B$  be types in a tropical pseudohyperplane arrangement  $\mathcal{T}$ . By Proposition 4.2 it suffices to show that  $S_{AB}$  is connected.



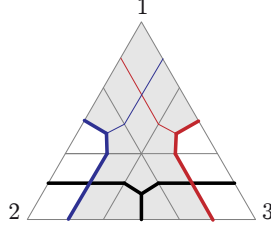


Figure 13: An approximating neighbourhood for  $a = 1, b = 23$  as an intersection of affine pseudohalfspaces in a blow-up of the black tropical pseudohyperplane.

By Lemma 6.4 we can approximate the set  $S_{AB} = \{C \mid C_i \in \{A_i, B_i, A_i \cup B_i\}\}$  as an intersection  $X = \bigcap H_i^+$  of pseudohalfspaces in an arrangement of affine pseudohyperplanes obtained by suitable blow-ups of  $S$ . By Proposition 5.4,  $X$  is connected.

Moreover,  $S_{AB}$  is homotopic to  $X$ . To see this we shrink the new tropical pseudohyperplanes, that were added during the blow-ups. Denote by  $S'$  the blow-up of  $S$  and assume without loss of generality that the original  $n$  tropical pseudohyperplanes have indices  $1, \dots, n$ . Moreover, assume that  $S'$  is a mixed subdivision of  $N\Delta^{d-1}$ . Consider the following homotopy:

$$H : [0, 1] \times S' \rightarrow S$$

$$\left( \lambda, \sum_{i=1}^N C_i \right) \mapsto \sum_{i=1}^n C_i + (1 - \lambda) \sum_{i=n+1}^N C_i.$$

It is clear that  $H$  is continuous. Moreover,  $H(X, 0) = X$  and  $H(X, 1) = S_{AB}$  and hence  $S_{AB}$  is homotopic to  $X$ .  $\square$

### 6.3 Non-fine mixed subdivisions

In this section we prove that arbitrary mixed subdivisions of  $n\Delta^{d-1}$  satisfy the elimination property.

We can still construct approximating neighbourhoods by means of blowing up tropical pseudohyperplanes even if the mixed subdivision is not fine.

The following is clear from the above:

**Lemma 6.8.** *Let  $S$  be a (not necessarily fine) mixed subdivision of  $n\Delta^{d-1}$ . Then  $\bigcup \{\mathcal{H}_i\}$  is an arrangement of affine pseudohyperplanes.*

With this we are now ready to prove the main result of this chapter:

**Theorem 6.9.** *Every mixed subdivision of  $n\Delta^{d-1}$  satisfies the elimination property.*

*Proof.* If we repeatedly blow-up  $S$  with respect to any  $(i, \pi, I)$  we obtain  $n(2^d - 2)d!$  new tropical pseudohyperplanes (one for each  $(i, \pi, I)$ ) and hence a mixed subdivision of  $(n + n(2^d - 2)d!)\Delta^{d-1}$ , in which we find our  $H_{I,p}$ s. It remains to show that these again form an arrangement of affine pseudohyperplanes. But this follows since if we

delete the  $n$  original tropical pseudohyperplanes we obtain a tropical pseudohyperplane arrangement in general position.

From here on the proof works as for Theorem 6.7: □

From this we immediately obtain the following corollaries:

**Corollary 6.10** ([AD09, Conjecture 5.1]). *Tropical oriented matroids with parameters  $(n, d)$  are in one-to-one correspondence with mixed subdivisions of  $n\Delta^{d-1}$  and subdivisions of  $\Delta^{n-1} \times \Delta^{d-1}$ .*

This completes the proof of the equivalence of the five concepts of tropical oriented matroids, tropical pseudohyperplane arrangements I ([H12b, Definition 4.3]) and II (Definition 5.5), mixed subdivisions of  $n\Delta^{d-1}$  and subdivisions of  $\Delta^{n-1} \times \Delta^{d-1}$  depicted in Figure 2.

Moreover, the duality relation between mixed subdivisions of  $n\Delta^{d-1}$  and  $d\Delta^{n-1}$  implies that the dual of a tropical oriented matroid is itself a tropical oriented matroid.

**Corollary 6.11** ([AD09, Conjecture 5.5]). *The dual of a tropical oriented matroid with parameters  $(n, d)$  is a tropical oriented matroid with parameters  $(d, n)$ .*

## References

- [AB07] Federico Ardila and Sara Billey. Flag arrangements and triangulations of products of simplices. *Adv. Math.* 214.2 (2007), pp. 495–524.
- [AD09] Federico Ardila and Mike Develin. Tropical hyperplane arrangements and oriented matroids. *Mathematische Zeitschrift* 262.4 (2009), pp. 795–816.
- [AK06] Federico Ardila and Caroline J. Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Combin. Theory Ser. B* 96.1 (2006), pp. 38–49.
- [BLS+99] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White and Günter M. Ziegler. *Oriented Matroids*. Cambridge University Press, 1999.
- [DRS10] Jesus A. De Loera, Jörg Rambau and Francisco Santos. *Triangulations: Applications, structures, algorithms. Algorithms and Computation in Mathematics*. Springer, 2010.
- [DS04] Mike Develin and Bernd Sturmfels. Tropical convexity. *Doc. Math.* 9 (2004), pp. 1–27.
- [FL78] Jon Folkman and Jim Lawrence. Oriented matroids. *J. Combin. Theory. Series B* 25.2 (1978), pp. 199–236.
- [GJ00] Evgenij Gawrilow and Michael Joswig. `polymake`: a Framework for Analyzing Convex Polytopes. In: *Polytopes — Combinatorics and Computation*. Ed. by Gil Kalai and Günter M. Ziegler. Birkhäuser, 2000, pp. 43–74.

- [Hoc72] Melvin Hochster. Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes. *The Annals of Mathematics*. Second Series 96.2 (1972), pp. 318–337.
- [H12a] Silke Horn. A Topological Representation Theorem for tropical oriented matroids. *DMTCS Proceedings* 01 (2012), pp. 135–146.
- [H12b] Silke Horn. A Topological Representation Theorem for tropical oriented matroids: Part I. *in preparation* (2012).
- [H12c] Silke Horn. tropmat: An extension for polymake. *Free software* <http://www.solros.de/polymake/tropmat> (2012).
- [H12d] Silke Horn. *Tropical Oriented Matroids and Cubical Complexes*. PhD Thesis, TU Darmstadt, 2012.
- [HRS00] Birkett Huber, Jörg Rambau and Francisco Santos. The Cayley trick, lifting subdivisions and the Bohne-Dress theorem on zonotopal tilings. *Journal of the European Mathematical Society* 2.2 (2000), pp. 179–198.
- [Mik06] Grigory Mikhalkin. Tropical geometry and its applications. In: *International Congress of Mathematicians. Vol. II*. Eur. Math. Soc., Zürich, 2006, pp. 827–852.
- [OY11] Suho Oh and Hwanchul Yoo. Triangulations of  $\Delta_{n-1} \times \Delta_{d-1}$  and tropical oriented matroids. *DMTCS Proceedings* 01 (2011), pp. 717–728.
- [San05a] Francisco Santos. Non-connected toric Hilbert schemes. *Math. Ann.* 332.3 (2005), pp. 645–665.
- [San05b] Francisco Santos. The Cayley trick and triangulations of products of simplices. In: *Integer points in polyhedra—geometry, number theory, algebra, optimization*. Vol. 374. Contemp. Math. Providence, RI: Amer. Math. Soc., 2005, pp. 151–177.
- [Zee63] Eric-Christopher Zeeman. Seminar on Combinatorial Topology. *Institut des Hautes Études Scientifiques* Fascicule 1.Exposés I à V inclus (1963).